TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 229, 1977

DISTRIBUTION OF EIGENVALUES IN THE PRESENCE OF HIGHER ORDER TURNING POINTS

BY
ANTHONY LEUNG[1]

ABSTRACT. This article is concerned with the eigenvalue problem $u''(x) - \lambda^2 p(x)u(x) = 0$, $u(x) \in L_2(-\infty, \infty)$, where $p(x)$ is real, analytic and possesses zeroes of arbitrary orders. Under certain conditions on $p(x)$, approximate formulas for the eigenvalues are found. The problem considered is of interest in the study of particle scattering and wave mechanics. The formula is analogous to the quantum rule of Bohr-Sommerfeld.

1. **Introduction.** This article is concerned with the eigenvalue problem

$$u''(x) - \lambda^2 p(x)u(x) = 0, \tag{1.1}$$

$$u \in L_2(-\infty, \infty) \tag{1.2}$$

where $p(x)$ is real analytic for $x \in (-\infty, \infty)$, and satisfies certain growth conditions as $x \to \pm\infty$. Problems of this type are of interest in the study of particle scattering and wave mechanics. Sibuya [4], Weinberg [6], and Evgrafov and Fedoryuk [1] have studied the asymptotic distribution of large positive eigenvalues for problems (1.1), (1.2). However, essentially they all assumed that $p(x)$ can only have pairs of first order zeroes on the real axis. It is not clear whether the approximation formulas for the large eigenvalues remain unaltered or significantly modified when higher, even and odd order zeroes of $p(x)$ are present on the real axis. The recent result by Leung [3] enables this investigation.

We make the following hypotheses on $p(x)$:

(H1) $p(x)$ is real analytic function for $x \in (-\infty, +\infty)$;

(H2a) $p(x)$ possesses a finite number of real zeroes $a_n < a_{n-1} < \cdots < a_1$ such that if we let α_i be the order of zero of $p(x)$ at $x = a_i$, $i = 1, \ldots, n$, then: α_2 is odd if α_1 is odd; for $k \geq 2$, α_{k+1} is odd if both conditions (i) α_k is odd and (ii) $\sum_{i=1}^{k-1}\alpha_i$ is even are satisfied.

(H2b) The sum of the order of zeroes, $\sum_{i=1}^{n}\alpha_i$, is even;

(H3) $p(x) = c_0 x^l[1 + q(x)]$ with $c_0 > 0$, $l \geq 0$ an integer, and

Received by the editors August 8, 1975.

AMS (MOS) subject classifications (1970). Primary 34E20, 34B25.

Key words and phrases. Linear differential equations, boundary value problem, eigenvalues, turning points, connection formulas.

([1]) This research was partially supported by the Faculty Research Fellowship, University of Cincinnati.

$$\lim_{|x|\to\infty} q(x) = 0;$$

(H4) $\int_{\bar{x}}^{+\infty}|\omega(x)|\,dx < \infty$, $\int_{-\infty}^{\bar{\bar{x}}}|\omega(x)|\,dx < \infty$, and $|p'(x)|\,|p(x)|^{-3/2}$ is bounded in $(-\infty, \bar{\bar{x}}) \cup (\bar{x}, \infty)$, where

$$\omega(x) = \frac{1}{8}\,\frac{p''(x)}{(p(x))^{3/2}} - \frac{5}{32}\,\frac{(p'(x))^2}{(p(x))^{5/2}}$$

and $-\infty < \bar{\bar{x}} < a_n < a_1 < \bar{x} < \infty$.

The function $p(x) = (x + 10)(x + 2)(x - 1)^2(x - 4)(x - 5)(x - 10)^2$ is a simple example which satisfies all the above conditions. On the other hand, the function $p(x) = (x + 10)(x + 2)^2(x - 1)(x - 4)(x - 5)(x - 10)^2$ fails to satisfy hypothesis (H2a).

Observe that hypotheses (H2a), (H2b) are significant generalizations of (H2) in [6]. (H4) is not assumed in [6], and can apparently be removed with slightly additional work and alteration in §3. The main result is stated below in Theorem 1.1, which is almost in exact analogy with that in [6]. Unfortunately, the presence of higher order zeroes of $p(x)$ allowed by (H2a) necessitates more complicated and different treatment from that in [6].

Solutions with known approximate formulas in regions bounded away from the turning points are constructed and continued along the real axis from $+\infty$ to $-\infty$, with their connections made around the zeroes of $p(x)$ through the use of the complex plane. All these connection formulas are carefully organized to derive a very simple equation which the eigenvalues must satisfy. This finally leads to estimates for large positive eigenvalues, λ, for problem (1.1), (1.2). In the process, connection methods and formulas by Evgrafov and Fedoryuk [1], and Leung [3] are used.

REMARKS. Hypotheses (H1) to (H3) imply that (i) $p(x) > 0$ for $x \in (-\infty, a_n) \cup (a_1, \infty)$; (ii) if $p(x) < 0$ for $x \in (a_{k+1}, a_k)$, then α_{k+1} and α_k are both odd; (iii) if odd order zeroes are present, such zeroes occur in adjacent pairs along the real axis.

Let

$$J = \{1 \leq j < n \,|\, p(x) < 0 \text{ for } x \in (a_{j+1}, a_j)\},$$
$$S = \{\lambda \in C \,|\, \operatorname{Re}\lambda \geq K_2, |\operatorname{Im}\lambda| \leq K_1\},$$

where K_1, K_2 are positive constants.

THEOREM 1.1. *The eigenvalues of problem* (1.1), (1.2) *satisfy the following properties*:

(i) *All eigenvalues in S of sufficiently large real parts are real.*

(ii) *As λ tends to infinity in S, all eigenvalues in S must satisfy the equation*

$$\prod_{\substack{k=1\\k\in J}}^{n-1}\left[1+\exp\left\{2\lambda\int_{a_{k+1}}^{a_k}\sqrt{p(x)}\,dx\right\}+O\left(\lambda^{-1/\gamma(k)}\right)\right]=\mu(\lambda) \tag{1.3}$$

where $\gamma(k)=\max\{\alpha_k,\ \alpha_{k+1}\}+2$, *and* $\mu(\lambda)$ *denotes a generic term for a quantity which is asymptotic to zero as* $\lambda\to\infty$ *in* S.

(iii) *Let* $2r$ *be the total number of odd order turning points on the real axis. Suppose* $r=0$, *then there is no eigenvalue of large modulus in* S. *Suppose* $r\neq 0$, *let* $\theta=\min\{1/(\alpha_k+2)\colon\ k=1,2,\ldots,n,\ \alpha_k\ odd\}$. *Then all large enough positive eigenvalues can be approximated by the following r sequences*:

$$\lambda_{k,\nu}=\frac{\left(\nu-\frac{1}{2}\right)\pi}{\int_{a_{k+1}}^{a_k}\sqrt{|p(x)|}\,dx}+O\left(\nu^{-\theta/r+\varepsilon}\right) \tag{1.4}$$

$k\in J,\ \nu=1,2,\ldots$. *Here* ε *is any positive small constant.*

REMARKS. The choice of $\sqrt{p(x)}$ in (1.3) will be specified at the end of §2. Hypothesis (H2a) implies that the number of odd order turning points on the real axis is even. The approximations in (1.4) may be formulated differently with more or less careful application of Rouché's Theorem to equation (1.3).

2. **Choice of variable and construction of neighborhoods.** The real zeroes of $p(x)$ are known as turning points, and the orders of the zeroes are known as the orders of the turning points. For each $k=1,2,\ldots,n$, $p(x)$ can be considered as defined in a circular neighborhood B_k in the complex plane about the point a_k (where $B_i\cap B_j=\varnothing$, $i\neq j$). In each B_k, there are α_k+2 analytic curves starting at a_k, along which Re $\int_{a_k}^{x}\sqrt{p(z)}\,dz=0$. These are known as Stokes curves, and we may assume that the α_k+2 Stokes curves start at a_k, end at the boundary of B_k, and never intersect each other except at a_k. Label the α_k+2 Stokes curves with two systems of conventions for subsequent convenience. On the upper open half-plane, denote l_{k1} (and $\tilde{l}_{k1}$) as the Stokes curve whose tangent line at a_k makes the least (and greatest) angle with the ray $[a_k,\infty)$ on the upper half-plane. Successively label the Stokes curves around a_k in a counterclockwise (and clockwise) direction as $l_{k2},\ldots,l_{k\alpha_k+2}$ (and $\tilde{l}_{k2},\ldots,\tilde{l}_{k\alpha_k+2}$). Let D_{kj} (and $\tilde{D}_{kj}$) be the simply connected domain in B_k bounded between l_{kj-1} and l_{kj} ($\tilde{l}_{kj-1}$ and $\tilde{l}_{kj}$). Here, let $l_{kj}=l_{ki}$ and $\tilde{l}_{kj}=\tilde{l}_{ki}$ if $j=i$ (mod α_k+2); and for later convenience also, let $D_{kj}=D_{ki}$, $\tilde{D}_{kj}=\tilde{D}_{ki}$ if $j=i$ (mod α_k+2). $p(x)$ is analytic in a neighborhood H of the interval $[a_n,a_1]$ in the complex plane. We may assume all the Stokes curves which are not horizontal in B_k intersect the boundary of H. In H, remove one Stokes curve at each a_k according to the following rules:

(i) Delete $\tilde{l}_{j+1,(\alpha_j+5)/2}$ and $l_{j,(\alpha_j+5)/2}$ whenever $p(x)$ is negative in an interval (a_{j+1},a_j).

(ii) Delete l_{j,α_j+2} if α_j is even.

(Note that hypotheses (H1) to (H3) ensure that the above two rules designate one Stokes curve to be removed at each real turning point a_j, unambiguously.) Let $\hat{H}$ be H with these curves and turning points removed. For $x \in \hat{H}$, we can assume

$$I(x) \equiv \xi(a_1, x) = \int_{a_1}^{x} \sqrt{p(z)}\, dz \tag{2.1}$$

to be a single-valued analytic function, with p chosen such that $\xi(a_1, x) > 0$ for real $x > a_1$.

In the following sections, we will construct various asymptotic solutions as $\lambda \to +\infty$. To facilitate these constructions, we define a subdomain Ω in $\hat{H}$. This subdomain Ω in the x-plane is defined as the preimage of certain sets in

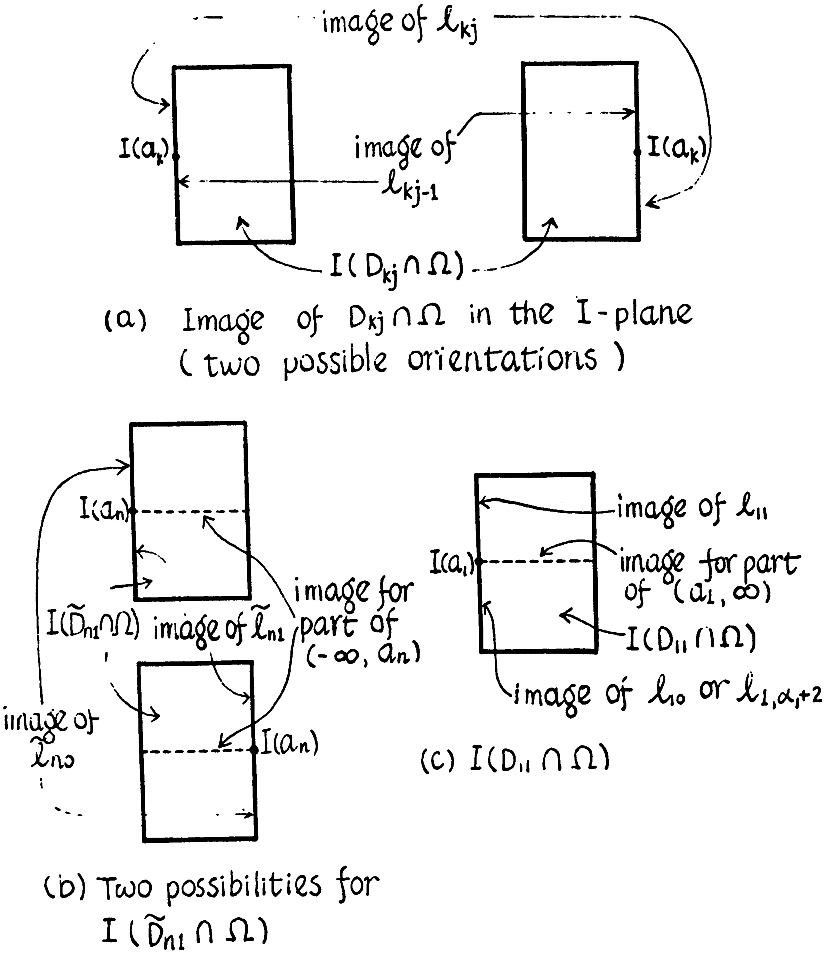

Figure 2.1

the I-plane, where $I = I(x) = \xi(a_1, x)$. Ω is to satisfy the following properties:

(1) If D_{kj} closure does not include a segment of the real axis, then $I(D_{kj} \cap \Omega)$ is a rectangle with $I(a_k)$ as midpoint of one of the vertical boundaries, $k = 1, \ldots, n$, $j = 1, 2, \ldots$; $I(D_{11} \cap \Omega)$ and $I(D_{n1} \cap \Omega)$ are rectangles with $I(a_1)$ and $I(a_n)$ as midpoints on their vertical boundaries. (Note that $p(x) > 0$ for x in (a_1, ∞) and $(-\infty, a_n)$, thus these intervals do not include any Stokes curve and $D_{11} \cap (a_1, \infty) \neq \varnothing$, $D_{n1} \cap (-\infty, a_n) \neq \varnothing$.) See Figure 2.1.

(2) Suppose $p(x) < 0$ on (a_{k+1}, a_k), let $H_{k+1,k}$ be the subdomain in $\hat{H}$ containing (a_{k+1}, a_k) and bounded by the four Stokes curves $\tilde{l}_{k+1,(\alpha_{k+1}+1)/2}$, $\tilde{l}_{k+1,(\alpha_{k+1}+5)/2}$, $l_{k,(\alpha_k+1)/2}$ and $l_{k,(\alpha_k+5)/2}$. Then $I(H_{k+1,k} \cap \Omega)$ is a rectangle with vertical and horizontal sides, and with two vertical line segments ending at $I(a_k)$, $I(a_{k+1})$ deleted. (See Figure 2.2.) $I(a_k)$, $I(a_{k+1})$ are equidistant from the two vertical sides, and are also of the same distance from their corresponding nearest horizontal sides. (Observe that $\mathrm{Re}\int_{a_k}^{x}\sqrt{p(z)}\,dz = 0$ for $x \in (a_{k+1}, a_k)$ in this situation.)

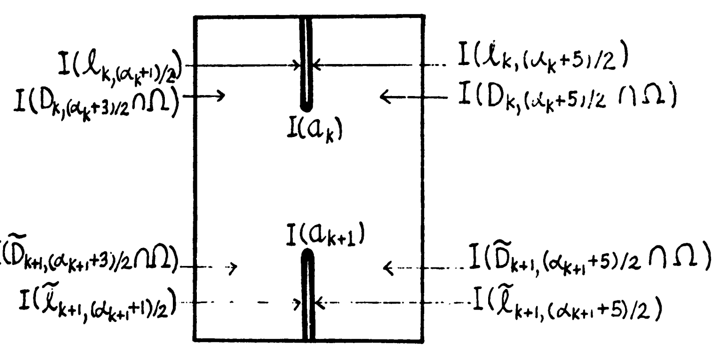

Figure 2.2 Image of $H_{k+1,k} \cap \Omega$ in the I-plane when $p(x) < 0$ in (a_{k+1}, a_k) (the case when $I(a_k)$ is above $I(a_{k+1})$)

(3) Suppose $p(x) > 0$ on (a_{k+1}, a_k), let $H_{k+1,k}$ be the subdomain in H containing (a_{k+1}, a_k) and bounded by the four Stokes curves $l_{k+1,1}$, $l_{k+1,0}$, $\tilde{l}_{k1}$ and $\tilde{l}_{k0}$. Then $I(H_{k+1,k} \cap \Omega)$ is a rectangle with $I(a_k)$ and $I(a_{k+1})$ as midpoints of the two vertical sides. (See Figure 2.3.)

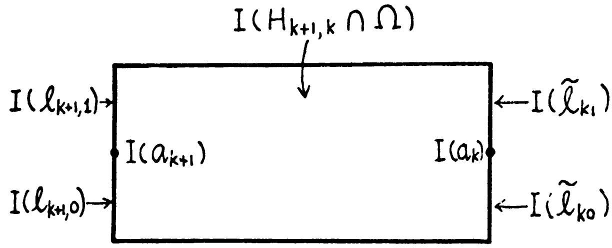

Figure 2.3 Image of $H_{k+1,k} \cap \Omega$ in the I-plane when $p(x) > 0$ in (a_{k+1}, k) (the case when $I(a_k)$ is on the right of $I(a_{k+1})$)

(4) All the heights of the rectangles in (1) and (3) are the same, and are equal to twice the distance of $I(a_k)$ to its nearest horizontal side in each of the rectangles in (2).

(5) All the widths of the rectangles in (1) are the same, and are equal to half of the widths of each rectangle in (2).

As long as the heights and widths in (4) and (5) are chosen and fixed to be sufficiently small, Ω will be an unambiguously defined subdomain of $\hat{H}$ after we include in Ω the parts of all the Stokes curves in $\hat{H}$ having the vertical sides of the rectangle in (1) and (3) as images. (It should be clear that in order to have Ω open, the preimages of all the sides of the rectangles in (1) to (3) should be excluded from Ω, if the side does not include image of a turning point.)

Remarks. (1) In all remaining parts of this article, in the connected region $(-\infty, a_n) \cup \Omega \cup (a_1, \infty)$, we will always choose $p(x)^{\pm 1/2}$ and $p(x)^{\pm 1/4}$ to be the unique continuations of the roots of p in the region, satisfying $p(x)^{\pm 1/2} > 0$ and $p(x)^{\pm 1/4} > 0$ in $\Omega \cap (a_1, \infty)$.

(2) For notational convenience, starting next section we will employ the following convention: suppose x, λ are restricted to certain definite regions, and a function $f(x)$ is defined there, the symbol $[f(x)]$ designates a quantity of the form $f(x)(1 + O(|\lambda|^{-1}))$ for all x, λ in the regions under consideration.

3. **Construction of asymptotic solutions as $x \to \pm\infty$.** The image in the I-plane of $\Omega \cap (D_{11} \cup l_{11} \cup D_{12})$ is a rectangle with vertical cut through the center as shown in Figure 3.1. Observe that the image of the horizontal line (a_1, ∞) under the mapping $I(x)$ and its continuation is precisely the right horizontal axis. Draw a sector in the rectangle with vertex at a point of distance δ vertically above $I(a_1) = 0$, central angle 2δ, and with angular

bisector coinciding with the cut. Choose $\delta > 0$ sufficiently small such that the vertex of the sector is inside the rectangle, and the sides of the sector intersect the bottom side of the rectangle at points closer to the midpoint of the bottom side than to the two lower vertices of the rectangle. Delete from $I(\Omega \cap (D_{11} \cup l_{11} \cup D_{12}))$ the closure of the sector constructed, and denote the preimage of this deleted rectangle in the x-plane by Λ_1.

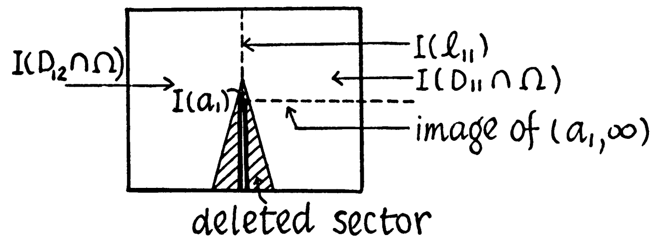

Figure 3.1 Image of $\Omega \cap (D_{11} \cup \ell_{11} \cup D_{12})$, and of Λ_1

LEMMA 3.1. *Equation* (1.1) *written in its corresponding matrix system*

$$\frac{dY}{dx} = \begin{bmatrix} 0 & 1 \\ \lambda^2 p(x) & 0 \end{bmatrix} Y, \qquad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{3.1}$$

has a fundamental matrix solution of the form

$$Y = Z_{12}^1(x, \lambda)$$

$$= \begin{bmatrix} [p(x)^{-1/4}]\exp\{-\lambda\xi(a_1, x)\} & [p(x)^{-1/4}]\exp\{\lambda\xi(a_1, x)\} \\ [-p(x)^{1/4}]\lambda\exp\{-\lambda\xi(a_1, x)\} & [p(x)^{1/4}]\lambda\exp\{\lambda\xi(a_1, x)\} \end{bmatrix} \tag{3.2}$$

as $|\lambda| \to \infty$ *in the region*

$$x \in \Lambda_1 \cup [a_1 + \varepsilon, \infty), \quad \textit{with } \varepsilon > 0 \textit{ and } I(a_1 + \varepsilon) \in I(\Lambda_1), \tag{3.3}$$

$$\lambda \in \{\lambda\colon \operatorname{Re}\lambda \geqslant K_2, |\operatorname{Im}\lambda| < K_1\} \tag{3.4}$$

where K_1, K_2 *are positive constants.* (*Differentiation is taken only along the real axis in the compliment of* Λ_1 *in* (3.3).)

Note that the first and second column of Z_{12}^1 are subdominant respectively in parts of D_{11} and D_{12}, with the subscript (i, j) of Z_{ij}^1 corresponding in order to the subscripts of D_{1i}, D_{1j}. We will consistently employ an analogous system of subscripts for other fundamental matrices.

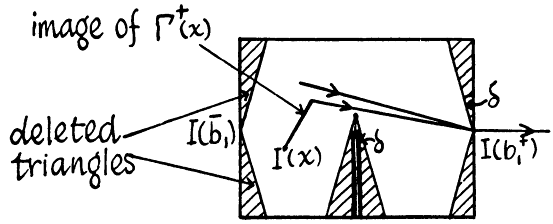

Figure 3.2 Q_1 and paths to $+\infty$

Proof. (Refer to Figure 3.2.) Denote the midpoints on the right and left vertical boundaries of the rectangle $I(\Omega \cap (D_{11} \cup l_{11} \cup D_{12}))$ as $I(b_1^+)$ and $I(b_1^-)$ respectively, and their corresponding preimages in the x-plane as b_1^+ and b_1^- respectively. At $I(b_1^+)$ draw two straight lines making angles δ with the vertical boundary, and delete the two closed, enclosed triangles inside the rectangle. Analogously delete two triangles on the left side of the rectangle. Let the resulting deleted region (i.e. with the sector and four triangles removed) union $I(b^+)$ be designated as Q_1, and its preimage in the x-plane by $I^{-1}(Q_1)$. At each point $I(x)$ in Q_1, one can easily draw a path in Q_1 from $I(x)$ to $I(b_1^+)$ such that the directed tangent vectors along the path always point to the right half-plane and always make angles $\geqslant \delta$ with the vertical. Extend the paths from $I(b_1^+)$ to $+\infty$ along the horizontal axis. Denote the preimage of each of these paths from $I(x)$ in Q_1 to $+\infty$ by $\Gamma^+(x)$. We now transform equation (3.1) into a Volterra integral equation such that the exponential term in the integrand is bounded along the path even as $\lambda \to \infty$ in region (3.4).

The transformation

$$(3.5)\quad Y = p(x)^{-1/4} \exp\{-\lambda\xi(a_1, x)\} \begin{bmatrix} 1 & 1 \\ \lambda\sqrt{p} - p'/4p & -\lambda\sqrt{p} - p'/4p \end{bmatrix} U$$

transform (3.1) into the system

$$(3.6)\qquad \frac{dU}{dx} = \left\{ \lambda\sqrt{p(x)} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \frac{\omega(x)}{\lambda} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \right\} U.$$

Recall that the choice of roots for $p(x)^{-1/4}$ and $\sqrt{p(x)}$ had been specified at the end of the previous section. Let $U(x) = \mathrm{col}(u_1, u_2)$; then a solution of the Volterra integral equations

$$(3.7)\qquad \begin{aligned} u_1(x) &= -\lambda^{-1}\int_{\Gamma^+(x)} \exp\{2\lambda\xi(t, x)\}\omega(t)(u_1 + u_2)\, dt, \\ u_2(x) &= 1 + \lambda^{-1}\int_{\Gamma^+(x)} \omega(t)(u_1 + u_2)\, dt \end{aligned}$$

will be a solution of (3.6) for $x \in I^{-1}(Q_1) \cup [b_1^+, \infty)$. Here $\Gamma^+(x)$ will simply be the path from x to $+\infty$ along the real axis when $x \in [b_1^+, \infty)$; $\xi(t, x) = \int_t^x \sqrt{p(z)}\, dz$ along $\Gamma^+(x)$. Because of the way the paths are constructed, we have Re $2\lambda\xi(t, x) \leqslant 0$ for $t \in \Gamma^+(x)$ and for $|\arg \lambda| < \delta$, which is true for sufficiently large $K_2 > 0$ in (3.4). If we write (3.7) in the form $U = U_0 + PU$, where $U_0 = \text{col}(0, 1)$ and P is the integral operator, we can show by induction that

$$\|P^n U_0\| \leqslant (2\omega^+ \lambda^{-1})^n, \qquad n = 0, 1, \ldots, \tag{3.8}$$

where the norm $\| \ \|$ is taken as the maximum of the absolute value of the two components and

$$\omega^+ = \sup_{x \in I^{-1}(Q_1) \cup [b_1^+, \infty)} \int_{\Gamma^+(x)} |\omega(t)|\, dt. \tag{3.9}$$

We have $\omega^+ < \infty$, by the first part of (H4). Therefore (3.8) implies that, for K_2 sufficiently large, the series $U = \sum_{n=0}^{\infty} P^n U_0$ converges uniformly and absolutely to a solution of (3.7) for $x \in I^{-1}(Q_1) \cup [b_1^+, \infty)$ and λ in (3.4). Further, (3.8) implies that $\|U\| \leqslant 2$ in this region. Expressing back in terms of Y by means of (3.5), and making use of the hypothesis concerning $|p'(x)|\, |p(x)|^{-3/2}$ in (H4), we arrive at a vector solution for (3.1) of the form displayed at the first column of (3.2).

To construct a vector solution of (3.1) of the form displayed in the second column of (3.2), change the factor $\exp\{-\lambda\xi(a_1, x)\}$ to $\exp\{\lambda\xi(a_1, x)\}$ in the transformation (3.5). The new transformed system corresponding to (3.6) would have the diagonal matrix $\text{diag}(0, -2\lambda\sqrt{p(x)})$ as the leading term of the coefficient matrix. Again, write the corresponding Volterra integral equation as in (3.7); however we now integrate along paths from x to b_1^-, for $x \in I^{-1}(Q_1) \cup [b_1^+, \infty)$. Denote the paths as $\Gamma^-(x)$. The exponential factor in the integrand becomes $\exp\{-2\lambda\xi(t, x)\}$, whose real part will again be $\leqslant 0$ by constructing $\Gamma^-(x)$ analogous to $\Gamma^+(x)$, in the reverse direction.

Finally, reduce the width of the rectangle $I(\Lambda_1)$; rename the narrower rectangle with the sector deleted, and its preimage with the same symbols: $I(\Lambda_1)$, Λ_1 respectively. Then formula (3.2) becomes valid for x in region (3.3), λ in (3.4). This proves Lemma 3.1.

We next consider the image of $\Omega \cap (\tilde{D}_{n1} \cup \tilde{l}_{n1} \cup \tilde{D}_{n2})$ in the I-plane. Delete a sector enclosing the cut in the image rectangle as before. Denote the preimage of this deleted rectangle in the x-plane by Λ_n. Note that the line segment $I(\tilde{l}_{n1} \cap \Omega)$ may be above or below $I(\tilde{a}_n)$. In the first case, $I(\tilde{D}_{n1})$ is on the left of the line, and in the second case $I(\tilde{D}_{n1})$ is on the right. (Refer to Figure 3.3.) Note that hypotheses (H2b) and (H3) imply that $p(x) > 0$, for $x < a_n$, and therefore the image of $(-\infty, a_n)$, under the mapping $I(x)$ and its continuation, is part of a horizontal line. The following two lemmas can be proved in the same way as Lemma 3.1.

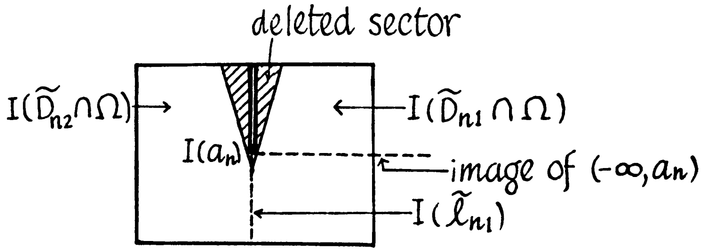

Figure 3.3 Image of $\Omega \cap (\tilde{D}_{n1} \cup \tilde{\ell}_{n1} \cup \tilde{D}_{n2})$ and of Λ_n (the case when $I(\tilde{\ell}_{n1} \cap \Omega)$ is below $I(a_n)$)

LEMMA 3.2. *Suppose* $I(\tilde{l}_{n1} \cap \Omega)$ *is above* $I(a_n)$, *then equation* (3.1) *has a fundamental matrix solution of the form*:

$$Y = \tilde{Z}^n_{21}(x, \lambda)$$

$$= \begin{bmatrix} [p(x)^{-1/4}] \exp\{-\lambda\xi(a_n, x)\} & [p(x)^{-1/4}] \exp\{\lambda\xi(a_n, x)\} \\ [-p(x)^{1/4}]\lambda \exp\{-\lambda\xi(a_n, x)\} & [p(x)^{1/4}]\lambda \exp\{\lambda\xi(a_n, x)\} \end{bmatrix} \tag{3.10}$$

as $|\lambda| \to \infty$ *in the region*: λ *in* (3.4) *and*

(3.11) $\quad x \in \Lambda_n \cup (-\infty, a_n - \varepsilon)$, *with* $\varepsilon > 0$ *and* $I(a_n - \varepsilon) \in I(\Lambda_n)$.

Here $\xi(a_n, x) = I(x) - I(a_n)$.

LEMMA 3.2*. *Suppose* $I(\tilde{l}_{n1} \cap \Omega)$ *is below* $I(a_n)$, *then equation* (3.1) *has a fundamental matrix solution* $Z^n_{21}(x, \lambda)$ *of the form displayed in* (3.10) *with the columns interchanged.*

REMARKS. We index the subscripts such that $\tilde{Z}^n_{ij}$ has its first column subdominant in $\tilde{D}_{ni}$ and second column subdominant in $\tilde{D}_{nj}$.

In §4 we will construct solutions with known approximate formulas close to or along the real axis between a_n and a_1. In §5 we will find the relationship between these various solutions by means of their formulas in overlapping regions of validity. Connecting these solutions with Z^1_{12} and $\tilde{Z}^n_{21}$ we will arrive at Theorem 5.1, which establishes the connection between Z^1_{12} and $\tilde{Z}^n_{21}$ explicitly.

4. **Construction of asymptotic solutions across turning points.** (a) For each $k = 1, \ldots, n$, we consider each pair of adjacent domains D_{kj}, D_{kj+1} which

does not have the Stokes curves l_{kj} between them cut away in constructing H. The image in the I-plane of $\Omega \cap (D_{kj} \cup l_{kj} \cup D_{kj+1})$ contains a rectangle with a vertical cut through the center, above or below $I(a_k)$. As in the beginning of §3, remove from the rectangle a narrow closed sector whose interior includes the vertical cut. The vertex of the sector is above or below $I(a_k)$ depending on whether the cut is below or above $I(a_k)$ respectively. Denote the preimage in the x-plane of the deleted rectangle by $\Lambda_k(j, j+1)$.

LEMMA 4.1. *Suppose* $I(l_{kj} \cap \Omega)$ *is above* $I(a_k)$, *then equation* (3.1) *has a fundamental matrix solution of the form*

$$Y = Y^k_{j,j+1}(x, \lambda)$$

$$(4.1)\qquad = \begin{bmatrix} [p(x)^{-1/4}]\exp\{-\lambda\xi(a_k, x)\} & [p(x)^{-1/4}]\exp\{\lambda\xi(a_k, x)\} \\ [-p(x)^{1/4}]\lambda\exp\{-\lambda\xi(a_k, x)\} & [p(x)^{1/4}]\lambda\exp\{\lambda\xi(a_k, x)\} \end{bmatrix}$$

as $|\lambda| \to \infty$ *in the region*

$$(4.2)\qquad x \in \Lambda_k(j, j+1),$$

$$(4.3)\qquad \lambda \in \{\lambda: \operatorname{Re}\lambda > K_2, |\operatorname{Im}\lambda| < K_1\}, \quad \textit{as in } (3.4).$$

Choose $\xi(a_k, x) = I(x) - I(a_k)$, $k = 1, \ldots, n$.

LEMMA 4.1*. *Suppose* $I(l_{kj} \cap \Omega)$ *is below* $I(a_k)$, *then equation* (3.1) *has a fundamental matrix solution* $Y^k_{j,j+1}(x, \lambda)$ *of the form displayed in* (4.1) *with the columns interchanged.*

The proof of the above two lemmas are almost exactly analogous to that of Lemma 3.1. The paths of integration will now end at the midpoints of the vertical boundaries of the rectangle $I(\Omega \cap (D_{kj} \cup l_{kj} \cup D_{kj+1}))$, instead of extending to infinity. Hypothesis (H4) is thus not required for the proofs.

(b) For each k such that $p(x) < 0$ in (a_{k+1}, a_k), $I(H_{k+1,k} \cap \Omega)$ is a rectangle with two disjoint vertical cuts of the form shown in Figure 2.2. We may assume that 2δ is smaller than the distance between $I(a_k)$ and $I(a_{k+1})$. At the point of δ distance vertically below the image of the turning point on the upper half of the rectangle, draw a sector of central angle 2δ enclosing the upper cut as before. Analogously draw a symmetric sector enclosing the lower cut. Delete the closures of these two sectors from $I(H_{k+1,k} \cap \Omega)$. Denote the preimage in the x-plane of this deleted rectangle by ${}_k\Lambda^-_{k+1}$. (See Figure 4.1.)

LEMMA 4.2. *Suppose* $I(a_k)$ *is above* $I(a_{k+1})$, *then equation* (3.1) *has fundamental matrix solutions*

$$Y = Z^k_{(\alpha_k+3)/2,(\alpha_k+5)/2}(x, \lambda) \quad \textit{and} \quad \tilde{Z}^{k+1}_{(\alpha_{k+1}+5)/2,(\alpha_{k+1}+3)/2}(x, \lambda),$$

respectively of the form:

$$\begin{bmatrix} [p(x)^{-1/4}]\exp\{\lambda\xi(a_k,x)\} & [p(x)^{-1/4}]\exp\{-\lambda\xi(a_k,x)\} \\ [p(x)^{1/4}]\lambda\exp\{\lambda\xi(a_k,x)\} & [-p(x)^{1/4}]\lambda\exp\{-\lambda\xi(a_k,x)\} \end{bmatrix}, \tag{4.4}$$

$$\begin{bmatrix} [p(x)^{-1/4}]\exp\{-\lambda\xi(a_{k+1},x)\} & [p(x)^{-1/4}]\exp\{\lambda\xi(a_{k+1},x)\} \\ [-p(x)^{1/4}]\lambda\exp\{-\lambda\xi(a_{k+1},x)\} & [p(x)^{1/4}]\lambda\exp\{\lambda\xi(a_{k+1},x)\} \end{bmatrix} \tag{4.5}$$

as $|\lambda| \to +\infty$, *for* λ *in region* (4.3) *and*

$$x \in {}_k\Lambda^-_{k+1}. \tag{4.6}$$

LEMMA 4.2*. *Suppose* $I(a_k)$ *is below* $I(a_{k+1})$, *then equation* (3.1) *has fundamental matrix solutions* $Z^k_{(\alpha_k+3)/2,(\alpha_k+5)/2}(x,\lambda)$ *and* $Z^{k+1}_{(\alpha_{k+1}+5)/2,(\alpha_{k+1}+3)/2}(x,\lambda)$ *respectively of the form* (4.4)* *and* (4.5)* *as* $|\lambda| \to \infty$ *in region* (4.3), (4.6). *Formulas* (4.4)* *and* (4.5)* *are respectively the matrices of the form* (4.4) *and* (4.5) *with adjacent columns interchanged.*

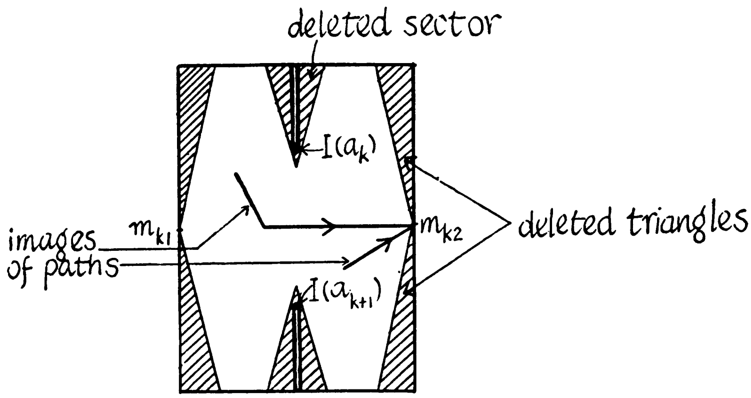

Figure 4.1 Image of ${}_k\Lambda^-_{k+1}$ and paths where $p(x)<0$ in (a_{k+1}, a_k) (the case when $I(a_k)$ is above $I(a_{k+1})$)

PROOF. On the midpoints m_{k1}, m_{k2} of the two vertical boundaries of the rectangle in Figure 4.1, draw lines making angle δ with the vertical sides. Remove the four enclosed triangles so formed inside the rectangle. At each point in the new deleted domain construct a path to m_{k2} such that the directed tangent vectors along the path always point to the right half-plane and make angles $\geqslant \delta$ with the vertical. Similarly construct paths to m_{k1}. Following the same procedure as in Lemma 3.1 with a_k and a_{k+1} successively replacing a_1, we will arrive at fundamental matrix solutions $Z^k_{(\alpha_k+3)/2,(\alpha_k+5)/2}$ and $Z^{k+1}_{(\alpha_{k+1}+5)/2,(\alpha_{k+1}+3)/2}$ with formulas (4.4) and (4.5) in the new deleted domain. As before, reduce the widths of the rectangle in Figure 4.1 and rename the narrower rectangle by the same symbol; formulas (4.4) and (4.5) will be valid for x in region (4.6).

(c) For each integer k such that $p(x) > 0$ in (a_{k+1}, a_k), $I(H_{k+1,k} \cap \Omega)$ is a rectangle of the form as shown in Figure 2.3. Consider the region

$$\left(D_{k+1,2} \cup l_{k+1,1} \cup H_{k+1,k} \cup \tilde{l}_{k1} \cup \tilde{D}_{k2}\right) \cap \Omega.$$

Its image in the I-plane has two rectangles of the form as shown in Figure 2.1, each attached to one side of $I(H_{k+1,k} \cap \Omega)$. (See Figure 4.2.) There are two cuts both directed in the same direction, above or below $I(a_k)$, $I(a_{k+1})$. Draw two sectors, with vertices above or below $I(a_k)$, $I(a_{k+1})$, enclosing the cuts in the usual manner. Delete the two closed sectors and denote the preimage in the x-plane of the deleted rectangle by ${}_k\Lambda_{k+1}^{+}$.

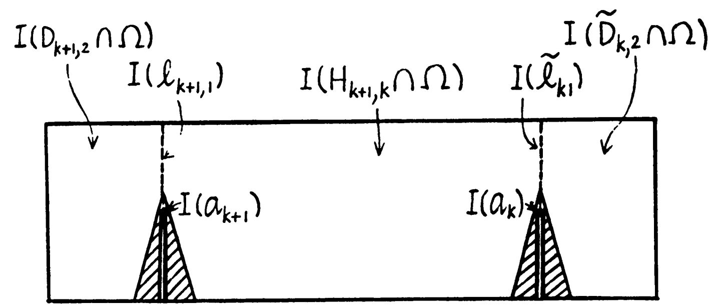

Figure 4.2. Image of ${}_k\Lambda_{k+1}^{+}$ where $p(x) > 0$ in (a_{k+1}, a_k) (the case where $I(a_k)$ is on the right of $I(a_{k+1})$, with cuts below them)

LEMMA 4.3. *Suppose* $I(\tilde{l}_{k1} \cap \Omega)$ *and* $I(l_{k+1,1} \cap \Omega)$ *are above* $I(a_k)$ *and* $I(a_{k+1})$ *respectively, then equation* (3.1) *has fundamental matrix solutions* $Y = \tilde{Z}_{21}^{k}(x, \lambda)$ *and* $Z_{12}^{k+1}(x, \lambda)$ *respectively of the forms*:

$$(4.7)\quad \begin{bmatrix} \left[p(x)^{-1/4}\right]\exp\{-\lambda\xi(a_k, x)\} & \left[p(x)^{-1/4}\right]\exp\{\lambda\xi(a_k, x)\} \\ \left[-p(x)^{1/4}\right]\lambda\exp\{-\lambda\xi(a_k, x)\} & \left[p(x)^{1/4}\right]\lambda\exp\{\lambda\xi(a_k, x)\} \end{bmatrix},$$

$$(4.8)\quad \begin{bmatrix} \left[p(x)^{-1/4}\right]\exp\{-\lambda\xi(a_{k+1}, x)\} & \left[p(x)^{-1/4}\right]\exp\{\lambda\xi(a_{k+1}, x)\} \\ \left[-p(x)^{1/4}\right]\lambda\exp\{-\lambda\xi(a_{k+1}, x)\} & \left[p(x)^{1/4}\right]\lambda\exp\{\lambda\xi(a_{k+1}, x)\} \end{bmatrix}$$

as $|\lambda| \to \infty$, *for* λ *in region* (4.3) *and*

(4.9) $$x \in {}_k\Lambda^+_{k+1}.$$

LEMMA 4.3*. *Suppose* $I(\tilde{l}_{k1} \cap \Omega)$ *and* $I(l_{k+1,1} \cap \Omega)$ *are below* $I(a_k)$ *and* $I(a_{k+1})$ *respectively, interchange the two columns in* (4.7) *as a formula for* $\tilde{Z}^k_{21}$, *and interchange the two columns in* (4.8) *as a formula for* Z^{k+1}_{12}.

The proofs are analogous to the previous lemmas, with the natural modifications.

REMARKS. Note that all the fundamental matrices have subscripts indexed in a consistent pattern. Namely, the first and second columns of Y^k_{ij} (or $\tilde{Y}^k_{ij}$; Z^k_{ij}; $\tilde{Z}^k_{ij}$) are subdominant respectively in D_{ki}, D_{kj} (or $\tilde{D}_{ki}$, $\tilde{D}_{kj}$; D_{ki}, D_{kj}; $\tilde{D}_{ki}$, $\tilde{D}_{kj}$). The order of the subscripts has direct relationship with the order of the regions in which the columns are subdominant. On the other hand, the superscript k relates to the subscript of the turning point a_k.

5. **Connections for different fundamental systems.** There are three basic types of connection matrices which are of essential importance to our problem:

(a) Connection matrix for $Y^k_{j,j+1}$ and $Y^k_{j+1,j+2}$ (when l_{kj} and l_{kj+1} are not removed from $\hat{H}$);

(b) Connection matrix for $Z^k_{(\alpha_k+3)/2,(\alpha_k+5)/2}$ and $\tilde{Z}^{k+1}_{(\alpha_{k+1}+5)/2,(\alpha_{k+1}+3)/2}$ (when $p(x) < 0$ in (a_{k+1}, a_k));

(c) Connection matrix for $\tilde{Z}^k_{21}$ and Z^{k+1}_{12} (when $p(x) > 0$ in (a_{k+1}, a_k)).

(a) The lemma in this subsection finds the relationship between fundamental systems of solutions with known approximations close to a fixed turning point. The asymptotic formulas for the two solutions of equation (1.1) displayed in the first row of (4.1) correspond to the first approximate of the formulas (2.10) in [3], where the equation $\varepsilon^2 u''(x) - p(x)u(x) = 0$ is considered. To adjust the notations, we identify ε^{-1} with our present λ, and observe that the index k here in the two articles plays different roles. By changing the variable locally around the turning point, "lateral" connection formulas for fundamental systems of solutions of form (4.1) are calculated in [3]. The fact that $p(x)$ is not a polynomial, in this present article, does not affect the calculation of the connection formula performed in [3]. As long as x is limited to a bounded neighborhood about the turning point, the crucial property for the successful completion of the connection procedure is that $p(x)$ is analytic. In terms of our present notations, [3] finds an approximate linear relation between two fundamental systems of solutions: one with the first and second column subdominant in D_{kj}, D_{kj+1} respectively as $\operatorname{Re}\lambda \to +\infty$, and the other with the first and second column subdominant in D_{kj+1}, D_{kj+2} respectively as $\operatorname{Re}\lambda \to +\infty$. (See formula (2.14) and Theorem 5.1 in [3].) When $\delta > 0$ is sufficiently small in §§3 and 4, formula (5.3) in [3] remains valid, after making the obvious change in notations. More precisely, the following is true.

LEMMA 5.1. *Let $k = 1, \ldots, n$. Suppose l_{kj}, l_{kj+1} are not removed from $\hat{H}$, then*

$$Y^k_{j+1,j+2}(x, \lambda) = Y^k_{jj+1}(x, \lambda) N_k(\lambda), \tag{5.1}$$

where

$$N_k(\lambda) = \begin{bmatrix} \mu(\lambda) & 1 + O(\lambda^{-1/(\alpha_k+2)}) \\ 1 + O(\lambda^{-1/(\alpha_k+2)}) & \dfrac{\omega_k^{-1} - \omega_k}{\omega_k^{-1} - 1}\,\omega_k^{-1-\alpha_k/4} + O(\lambda^{-1/(\alpha_k+2)}) \end{bmatrix} \tag{5.2}$$

as $\lambda \to \infty$ in region (3.4). *Here $\omega_k = \exp\{2\pi i/(\alpha_k + 2)\}$ and $\mu(\lambda)$ denotes a generic term for a quantity asymptotic to zero as $\lambda \to \infty$ in region* (3.4).

(b) The following two lemmas find the relationship between two fundamental systems of solutions, both with known approximation in the same open interval in which $p(x) < 0$.

LEMMA 5.2. *Suppose $p(x) < 0$ in (a_{k+1}, a_k) and $I(a_k)$ is above $I(a_{k+1})$, then*

$$\tilde{Z}^{k+1}_{(\alpha_{k+1}+5)/2,(\alpha_{k+1}+3)/2}(x, \lambda) = Z^k_{(\alpha_k+3)/2,(\alpha_k+5)/2}(x, \lambda) M_{k+1,k}(\lambda) \tag{5.3}$$

where

$$M_{k+1,k}(\lambda) = \begin{bmatrix} \mu(\lambda) & \exp\{2\lambda\xi(a_{k+1}, a_k)\}(1 + O(\lambda^{-1})) \\ 1 + O(\lambda^{-1}) & \mu(\lambda) \end{bmatrix} \cdot \exp\{-\lambda\xi(a_{k+1}, a_k)\} \tag{5.4}$$

as $\lambda \to \infty$ in region (3.4). *Here $\xi(a_{k+1}, a_k) = I(a_k) - I(a_{k+1})$.*

PROOF. $M_{k+1,k}(\lambda) = [Z^k_{(\alpha_k+3)/2,(\alpha_k+5)/2}]^{-1}[\tilde{Z}^{k+1}_{(\alpha_{k+1}+5)/2,(\alpha_{k+1}+3)/2}]$ which can be analyzed by using formulas (4.4) and (4.5). Straightforward computation gives

$$M_{k+1,1}(\lambda) = \frac{\exp\{-\lambda\xi(a_{k+1}, a_k)\}}{-2\lambda(1 + O(\lambda^{-1}))} \cdot \begin{bmatrix} \exp\{-2\lambda\xi(a_k, x)\}O(1) & \exp\{2\lambda\xi(a_{k+1}, a_k)\}(-2\lambda + O(1)) \\ -2\lambda + O(1) & \exp\{2\lambda\xi(a_{k+1}, a_k) + 2\lambda\xi(a_k, x)\}O(1) \end{bmatrix}. \tag{5.5}$$

The matrix in (5.5) should be independent of x, therefore its entries can be evaluated anywhere in the region (4.6). For an estimate of the entry on the upper left-hand corner, choose $\hat{x} \in {}_k\Lambda_{k+1}^-$ such that $\xi(a_k, \hat{x}) > 0$; thus the entry is asymptotic to zero at $\lambda \to \infty$ in (3.4). For an estimate of the entry on the lower right-hand corner, choose $\bar{x} \in {}_k\Lambda_{k+1}^-$ such that $\xi(a_k, \bar{x}) < 0$; and the entry will clearly be asymptotic to zero as $\lambda \to \infty$ in (3.4) too.

LEMMA 5.2* *Suppose $p(x) < 0$ in (a_{k+1}, a_k) and $I(a_k)$ is below $I(a_{k+1})$, then the two off-diagonal entries of the matrix on the right side of formula* (5.4) *should be interchanged for approximating $M_{k+1,k}(\lambda)$.*

REMARKS. Note that in the two lemmas above, the off diagonal terms of $M_{k+1,k}(\lambda)$ are bounded as $\lambda \to +\infty$ on the real axis.

(c) The following lemmas find the relationship between two fundamental systems of solutions, both with known approximation in the same open interval in which $p(x) > 0$.

LEMMA 5.3. *Suppose $p(x) > 0$ in (a_{k+1}, a_k), and $I(\tilde{l}_{k,1} \cap \Omega)$, $I(l_{k+1,1} \cap \Omega)$ are above $I(a_k)$, $I(a_{k+1})$ respectively, then*

$$Z_{12}^{k+1}(x, \lambda) = \tilde{Z}_{21}^{k}(x, \lambda) R_{k+1,k}(\lambda), \tag{5.6}$$

where

$$R_{k+1,k}(\lambda) = \exp\{\lambda\xi(a_{k+1}, a_k)\} \begin{bmatrix} \mu(\lambda) & \mu(\lambda) \\ \mu(\lambda) & 1 + O(\lambda^{-1}) \end{bmatrix} \tag{5.7}$$

as $\lambda \to \infty$ in region (3.4).

PROOF. Straightforward computation using (4.7) and (4.8) gives

$$R_{k+1,k}(\lambda) = \frac{\exp\{\lambda\xi(a_{k+1}, a_k)\}}{2\lambda(1 + O(\lambda^{-1}))}$$

$$\cdot \begin{bmatrix} \exp\{-2\lambda\xi(a_{k+1}, a_k)\}(2\lambda + O(1)) & \exp\{2\lambda\xi(a_k, x)\} O(1) \\ \exp\{-2\lambda\xi(a_{k+1}, a_k) - 2\lambda\xi(a_k, x)\} O(1) & 2\lambda + O(1) \end{bmatrix}$$

which should be independent of x. The term $\exp\{-2\lambda\xi(a_{k+1}, a_k)\}$ is clearly of the form $\mu(\lambda)$, since $\xi(a_{k+1}, a_k) > 0$. To evaluate the term on the upper right-hand corner, choose $\check{x} \in {}_k\Lambda_{k+1}^+$ such that $\xi(a_k, \check{x}) < 0$. To evaluate the term on the lower left-hand corner, choose $\tilde{\check{x}} \in {}_k\Lambda_{k+1}^+$ such that $\xi(a_k, \tilde{\check{x}}) > 0$.

LEMMA 5.3*. *Suppose $p(x) > 0$ in (a_{k+1}, a_k), and $I(\tilde{l}_{k1} \cap \Omega)$, $I(l_{k+1,1} \cap \Omega)$ are below $I(a_k)$, $I(a_{k+1})$ respectively, then the term $\exp\{\lambda\xi(a_{k+1}, a_k)\}$ on the right side of formula* (5.7) *should be replaced by $\exp\{\lambda\xi(a_k, a_{k+1})\}$. Formula* (5.6) *is unchanged.* (*Note: In this case $\xi(a_k, a_{k+1}) > 0$.*)

Lemmas 5.1 to 5.3 describe the three basic types of connection matrices listed in the beginning of this section. Eventually, we will connect $\tilde{Z}_{21}^{n}$ with Z_{12}^{1}. To facilitate the analysis of this final connection matrix, it is expedient first to investigate the structures of the following kinds of connection matrices, which are products of the first three basic types:

(I) Connection matrix between two fundamental systems, one with known approximation on the left of a_{k+1}, and one with known approximation on the right of a_k, when $p(x) < 0$ in (a_{k+1}, a_k). Refer to Lemma 5.4.

(II) Connection matrix between two fundamental systems, each with known approximation on left or right of a fixed even order turning point. Refer to Lemma 5.5.

LEMMA 5.4. *Suppose $p(x) < 0$ in (a_{k+1}, a_k), then*

$$\tilde{Z}_{21}^{k+1}(x,\lambda) = Z_{12}^{k}(x,\lambda)\left[\hat{N}_k(\lambda)M_{k+1,k}(\lambda)\hat{N}_{k+1}(\lambda)\right], \tag{5.8}$$

where

$$\hat{N}_k(\lambda) = \left[N_k(\lambda)\right]^{(\alpha_k+1)/2} \quad \text{and} \quad \hat{N}_{k+1}(\lambda) = \left[N_{k+1}(\lambda)\right]^{(\alpha_{k+1}+1)/2}. \tag{5.9}$$

Each (i,j)th entry, $1 \leqslant i, j \leqslant 2$, of the connection matrix $\hat{N}_k M_{k+1,k}\hat{N}_{k+1}$ in (5.8) *is of the form*:

$$\begin{aligned}&\left[g_{kij}(\lambda) + h_{kij}(\lambda)\exp\{2\lambda\xi(a_{k+1}, a_k)\} + \mu(\lambda)\right]\\ &\qquad\cdot\exp\{-\lambda\xi(a_{k+1}, a_k)\}\left[1 + O(\lambda^{-1})\right],\end{aligned} \tag{5.10}$$

as $\lambda \to \infty$ in region (3.4). *Here $g_{kij}(\lambda)$ and $h_{kij}(\lambda)$ are products of the entries of $\hat{N}_k$ and $\hat{N}_{k+1}$.*

If $I(a_k)$ is above $I(a_{k+1})$, then

$$g_{k22}(\lambda) = \rho_{k22}(\lambda)\rho_{k+1,12}(\lambda), \quad h_{k22}(\lambda) = \rho_{k21}(\lambda)\rho_{k+1,22}(\lambda) \tag{5.11}$$

where ρ_{kij}, $\rho_{k+1,ij}$ is the (i,j)th entry of $\hat{N}_k$ and $\hat{N}_{k+1}$ respectively.

If $I(a_k)$ is below $I(a_{k+1})$, then

$$g_{k22}(\lambda) = \rho_{k21}(\lambda)\rho_{k+1,22}(\lambda), \quad h_{k22}(\lambda) = \rho_{k22}(\lambda)\rho_{k+1,12}(\lambda). \tag{5.12}$$

PROOF. Since $p(x) < 0$ in (a_{k+1}, a_k), hypotheses (H1) to (H3) imply that α_{k+1}, α_k are both odd. Thus $\tilde{l}_{k+1,1_\sim} = l_{k+1,(\alpha_{k+1}+1)/2}$, $\tilde{l}_{k+1,j_\sim} = l_{k+1,(\alpha_{k+1}+1)/2-(j-1)}$, for $j = 1, 2, \ldots,$ and $D_{k+1,1} = D_{k+1,(\alpha_{k+1}+1)/2+1}$, $D_{k+1,j} = D_{k+1,(\alpha_{k+1}+1)/2+2-j}$, for $j = 1, 2, \ldots$. Let

$$\tilde{Y}_{j+1,j}^{k+1}(x,\lambda) = Y^{k+1}_{(\alpha_{k+1}+1)/2+2-(j+1),(\alpha_{k+1}+1)/2+2-j}, \qquad j = 1, \ldots, (\alpha_{k+1}+3)/2. \tag{5.13}$$

By (5.13) and Lemma 5.1, we have

$$
\begin{aligned}
\tilde{Y}^{k+1}_{j+1,j} &= Y^{k+1}_{(\alpha_{k+1}+1)/2-j+1,(\alpha_{k+1}+1)/2-j+2} \\
&= Y^{k+1}_{(\alpha_{k+1}+1)/2-j,(\alpha_{k+1}+1)/2-j+1} N_{k+1}(\lambda) \\
&= \tilde{Y}^{k+1}_{j+2,j+1} N_{k+1}(\lambda)
\end{aligned} \tag{5.14}
$$

for $j = 1, \ldots, (\alpha_{k+1} + 1)/2$. Successive applications of (5.14) for $j = 2, 3, \ldots, (\alpha_{k+1} - 1)/2$, gives

$$\tilde{Y}^{k+1}_{32} = \tilde{Y}^{k+1}_{(\alpha_{k+1}+3)/2,(\alpha_{k+1}+1)/2}[N_{k+1}]^{(\alpha_{k+1}-3)/2}. \tag{5.15}$$

When $j = 1$ in (5.14), $\tilde{Y}^{k+1}_{21} = Y^{k+1}_{(\alpha_{k+1}+1)/2,(\alpha_{k+1}+1)/2+1}$ has exactly the same approximate formulas as $Z^{k+1}_{21}(x, \lambda)$, for λ in region (3.4) and

$$x \in \Lambda_{k+1}((\alpha_{k+1} + 1)/2, (\alpha_{k+1} + 1)/2 + 1) \subset_{k+1}\Lambda^{+}_{k+2}.$$

(Refer to formulas (4.1) and (4.7), replacing all a_k by a_{k+1}; or refer alternatively to these same formulas with columns interchanged as described in Lemma 4.1* and Lemma 4.3*.) The connection between the two fundamental systems $Y^{k+1}_{(\alpha_{k+1}+1)/2,(\alpha_{k+1}+1)/2+1}$ and $Y^{k+1}_{(\alpha_{k+1}+1)/2-1,(\alpha_{k+1}+1)/2}$ in Lemma 5.1 are calculated through using their approximation formulas in the region of common validity. Thus $Z^{k+1}_{21}(x, \lambda)$ should be related to $Y^{k+1}_{(\alpha_{k+1}+1)/2-1,(\alpha_{k+1}+1)/2}$ (i.e. $\tilde{Y}^{k+1}_{32}$) by

$$\tilde{Z}^{k+1}_{21}(x, \lambda) = \tilde{Y}^{k+1}_{32}(x, \lambda) N_{k+1}(\lambda), \tag{5.16}$$

in the same way as $\tilde{Y}^{k+1}_{21}$ is related to $\tilde{Y}^{k+1}_{32}$. When $j = (\alpha_{k+1} + 1)/2$ in (5.14), analogous reason gives

$$\tilde{Y}^{k+1}_{(\alpha_{k+1}+3)/2,(\alpha_{k+1}+1)/2} = \tilde{Z}^{k+1}_{(\alpha_{k+1}+5)/2,(\alpha_{k+1}+3)/2} N_{k+1}. \tag{5.17}$$

Since $\tilde{Y}^{k+1}_{(\alpha_{k+1}+5)/2,(\alpha_{k+1}+3)/2}$ and $\tilde{Z}^{k+1}_{(\alpha_{k+1}+5)/2,(\alpha_{k+1}+3)/2}$ have the same approximation formulas in the appropriate region. Combining (5.15) to (5.17), we have

$$\tilde{Z}^{k+1}_{21} = \tilde{Z}^{k+1}_{(\alpha_{k+1}+5)/2,(\alpha_{k+1}+3)/2}[N_{k+1}]^{(\alpha_{k+1}+1)/2}. \tag{5.18}$$

Analogously,

$$Z^{k}_{(\alpha_k+3)/2,(\alpha_k+5)/2} = Z^{k}_{12}[N_k]^{(\alpha_k+1)/2}. \tag{5.19}$$

Formulas (5.18), (5.3) and (5.19) lead to formulas (5.8) and (5.9) in the statement of this lemma. Finally, formulas (5.10) to (5.12) are simply direct computation from (5.8), using the approximation formula for $M_{k+1,k}(\lambda)$ in (5.4).

LEMMA 5.5. *Suppose a_k is an even order turning point, then*

$$\tilde{Z}^{k}_{21}(x, \lambda) = Z^{k}_{12}(x, \lambda)[N_k(\lambda)]^{\alpha_k/2}. \tag{5.20}$$

Note that hypotheses (H1) to (H3) imply that $p(x) > 0$ immediately to the left and right of $x = a_k$, when α_k is even.

PROOF. The proof is analogous to the procedures in the last lemma from the beginning to formula (5.18).

In view of statements (I), (II), one sees that Lemmas 5.4, 5.5, together with Lemmas 5.3 and 5.3*, provide sufficient background for continuation of solution for $+\infty$ to $-\infty$. For notational purpose, we partition the turning points $a_1, \ldots, a_n$ into m pairwise nonintersecting sets of points $G_1, \ldots, G_m$, where $0 < m \leqslant n$. Each G_i is to contain a pair of successive turning points of odd order or a single turning point of even order, grouped according to the following manner: G_1 contains a_1 if α_1 is even, or contains a_1, a_2 if α_1 is odd (note that our hypotheses imply that α_2 is odd if α_1 is odd); inductively, if the largest turning point a_j less than the turning point(s) in G_{k-1} is of even order, then G_k contains a_j only, otherwise G_k contains a_j and a_{j+1}. (Observe that hypotheses (H2a), (H2b) insure that the selection process is unambiguously defined to a final step m.) Let

$$(5.21) \qquad T_j(\lambda) = \begin{cases} \hat{N}_{\phi(j)} M_{\phi(j)+1,\phi(j)} \hat{N}_{\phi(j)+1}, & \text{if } G_j \text{ has two turning points;} \\ [N_{\psi(j)}]^{\alpha_{\psi(j)}/2}, & \text{if } G_j \text{ has one turning point} \end{cases}$$

for $j = 1, \ldots, m$, where $\phi(j)$ is the subscript of the larger turning point in G_j and $\psi(j)$ is the subscript of the turning point in G_j.

THEOREM 5.1.

$$(5.22) \qquad \tilde{Z}_{21}^n = Z_{12}^1 \left[T_1 \prod_{j=1}^{m-1} (R_j T_{j+1})(\lambda) \right]$$

for $\lambda \in \{\lambda\colon \operatorname{Re} \lambda > K_2, |\operatorname{Im} \lambda| \leqslant K_1\}$, K_1, K_2 *are positive constants*, K_2 *dependent of* K_1. *Here*

$$(5.23) \qquad R_j = \begin{cases} R_{\phi(j)+2,\phi(j)+1}(\lambda) & \textit{if } G_j \textit{ has two turning points;} \\ R_{\psi(j)+1,\psi(j)}(\lambda) & \textit{if } G_j \textit{ has one turning point.} \end{cases}$$

PROOF. This is a direct application of Lemmas 5.3 or 5.3*, 5.4 and 5.5.

6. **Estimates for large positive eigenvalues.** We will estimate the distribution of eigenvalues λ as $\lambda \to \infty$ in region (3.4). This will be done by analyzing carefully the connection matrix $T_1 \prod_{j=1}^{m-1} (R_j T_{j+1})$ for the two fundamental systems $\tilde{Z}_{21}^n$ and Z_{12}^1 given in Theorem 5.1.

LEMMA 6.1. *If* G_{j+1} *has two turning points*, $m-1 \geqslant j \geqslant 1$, *then, as* $\lambda \to \infty$ *in region* (3.4), *the matrix* $R_j(\lambda) T_{j+1}(\lambda)$ *has the form*:

$$(6.1) \quad \exp\{\pm\lambda\xi(a_{\bar{\rho}(j)+1}, a_{\bar{\rho}(j)})\} \exp\{-\lambda\xi(a_{\phi(j+1)+1}, a_{\phi(j+1)})\} \begin{bmatrix} \mu(\lambda) & \mu(\lambda) \\ v_1^j(\lambda) & v_2^j(\lambda) \end{bmatrix}$$

where

$$\bar{\rho}(j) = \begin{cases} \phi(j) + 1 & \text{if } G_j \text{ has two turning points;} \\ \psi(j) & \text{if } G_j \text{ has one turning point,} \end{cases}$$

and v_k^j, $k = 1, 2$, *are of the form*

(6.2) $$\left[p_k^j(\lambda) + q_k^j(\lambda)\exp\{2\lambda\xi(a_{\phi(j+1)+1}, a_{\phi(j+1)})\} + \mu(\lambda)\right]\left[1 + O(\lambda^{-1})\right].$$

Here $p_k^j(\lambda)$, $q_k^j(\lambda)$ *are products of the entries of* $\hat{N}_{\phi(j+1)}$ *and* $\hat{N}_{\phi(j+1)+1}$. *In particular*

(6.3) $$p_2^j(\lambda) = \rho_{\phi(j+1),2,2}\rho_{\phi(j+1)+1,1,2}, \qquad q_2^j(\lambda) = \rho_{\phi(j+1),2,1}\rho_{\phi(j+1)+1,2,2},$$

or interchange p_2^j *and* q_2^j *on the left side of* (6.3). (*The* ρ*'s in* (6.3) *are described in Lemma* 5.4.)

PROOF. Apply Lemma 5.3 (or Lemma 5.3*) and (5.10) to (5.12). Observe that $\xi(a_{\phi(j+1)+1}, a_{\phi(j+1)})$, which corresponds to $\xi(a_{k+1}, a_k)$ in (5.10), is pure imaginary. Note that in the first exponential term of (6.1), the + sign is used when Lemma 5.3 is applied, and − sign is used when Lemma 5.3* is applicable.

LEMMA 6.2. *If* G_{j+1} *has one turning point* $m - 1 \geqslant j \geqslant 1$, *then, as* $\lambda \to \infty$ *in region* (3.4), *the matrix* $R_j(\lambda)T_{j+1}(\lambda)$ *has the form*

(6.4) $$\exp\{\pm\lambda\xi(a_{\bar{\rho}(j)+1}, a_{\bar{\rho}(j)})\}\begin{bmatrix} \mu(\lambda) & \mu(\lambda) \\ v_1^j(\lambda) & v_2^j(\lambda) \end{bmatrix},$$

where

(6.5) $$v_k^j(\lambda) = \beta_{\psi(j+1),2,k}\left[1 + O(\lambda^{-1})\right], \qquad k = 1, 2,$$

and $\beta_{\psi(j+1),2,k}$ *is the* (2, k)*th entry of the matrix* $[N_{\psi(j+1)}(\lambda)]^{\alpha_{\psi(j+1)}/2}$.

PROOF. Simply apply (5.2) and Lemma 5.3 (or Lemma 5.3*). The choice of ± sign in the exponential term in (6.4) is determined in the same way as in the last lemma.

Clearly $v_2^j(\lambda)$, $j = 1, \ldots, m - 1$, is of great significance in determining the lower right-hand corner entry of the connecting matrix $T_1\prod_{j=1}^{m-1}R_jT_{j+1}$. The following two lemmas compute them carefully. First, introduce a few more necessary symbols. When G_r has two turning points, let

(6.6) $$\theta_{r1} = 2\pi/(\alpha_{\phi(r)} + 2), \qquad \theta_{r2} = 2\pi/(\alpha_{\phi(r)+1} + 2).$$

When G_r has one turning point, let

(6.7) $$\theta_r = 2\pi/(\alpha_{\psi(r)} + 2).$$

LEMMA 6.3. *If* G_{j+1} *has one turning point, then the number* $\beta_{\psi(j+1),2,2}$ *in* (6.5) *is given by the formula*

$$(6.8)\quad \begin{aligned}\beta_{\psi(j+1),2,2}(\lambda) &= \left[(-i)^{1+(\alpha_{\psi(j+1)})/2}/\sin(\theta_{j+1}/2)\right] \\ &\quad \cdot\left[\operatorname{Im}\exp\left\{i\theta_{j+1}\left(1+\tfrac{1}{2}\alpha_{\psi(j+1)}\right)/2\right\}\right]i + O\left(\lambda^{-1/(\alpha_{\psi(j+1)}+2)}\right),\end{aligned}$$

as $\lambda \to \infty$ *in region* (3.4). *The leading term on the right side of equation* (6.8) *is nonzero, independent of* λ.

PROOF. Let the (2, 2)th entry of the matrix $[N_{\psi(j+1)}]^k$ be denoted by c_k, $k = 0, 1, \ldots, \alpha_{\psi(j+1)}/2$. Without essential difficulty, we can use formula (5.2) to deduce that c_k's satisfy the recursive relations

$$(6.9)\quad \begin{aligned} c_0 &= 1,\\ c_1 &= \frac{\omega_{\psi(j+1)}^{-1} - \omega_{\psi(j+1)}}{\omega_{\psi(j+1)}^{-1} - 1}\,\omega_{\psi(j+1)}^{-1-\alpha_{\psi(j+1)}/4} + O\left(\lambda^{-1/(\alpha_{\psi(j+1)}+2)}\right)\\ &= -2\cos\left(\frac{\theta_{j+1}}{2}\right)i + O\left(\lambda^{-1/(\alpha_{\psi(j+1)}+2)}\right),\\ c_{k+2} &= -2\cos\left(\frac{\theta_{j+1}}{2}\right)ic_{k+1} + c_k + O\left(\lambda^{-1/(\alpha_{\psi(j+1)}+2)}\right)\end{aligned}$$

for $k = 0, 1, \ldots, \alpha_{\psi(j+1)}/2 - 2$. By standard methods of difference equations, the solution of the difference equation

$$(6.10)\quad b_0 = 1,\quad b_1 = -2\cos\left(\frac{\theta_{j+1}}{2}\right)i,\quad b_{k+2} = -2\cos\left(\frac{\theta_{j+1}}{2}\right)ib_{k+1} + b_k,$$

$k = 0, \ldots, \alpha_{\psi(j+1)}/2 - 2$, is given by

$$(6.11)\quad b_k = \frac{(-1)^{k+1}i^{k+1}}{2\sin(\theta_{j+1}/2)}\left[\exp\left\{i\frac{\theta_{j+1}}{2}(k+1)\right\} - \exp\left\{-i\frac{\theta_{j+1}}{2}(k+1)\right\}\right].$$

One easily checks that $b_k \neq 0$ when $k = 0, 1, \ldots, \alpha_{\psi(j+1)}$ and $b_{\psi(j+1)+1} = 0$ in formula (6.11). Comparing (6.9) and (6.10), one sees that $c_k = b_k + O(\lambda^{-1/(\alpha_{\psi(j+1)}+2)})$. Therefore $c_k \neq 0$, for $k = 0, \ldots, \alpha_{\psi(j+1)}$, and large $|\lambda|$. In particular, $\beta_{\psi(j+1),2,2}(\lambda) = c_{\alpha_{\psi(j+1)}/2}$. This proves Lemma 6.3.

LEMMA 6.4. *If* G_{j+1} *has two turning points, then* $\rho_{\phi(j+1),2,k}$, $\rho_{\phi(j+1)+1,k,2}$, $k = 1, 2$, *in* (6.3) *is given by the formulas*

$$\rho_{\phi(j+1),2,k}(\lambda) = (-i)^{(\alpha_{\phi(j+1)}+1)/2+k-1} \cdot \left[\sin\left(\frac{\theta_{j+1,1}}{2}\right)\right]^{-1}$$

$$\cdot i \operatorname{Im} \exp\left\{ i \frac{\theta_{j+1,1}}{2} \left[\tfrac{1}{2}(\alpha_{\phi(j+1)} + 1) + k - 1 \right] \right\}$$

$$+ O(\lambda^{-1/(\alpha_{\phi(j+1)}+2)}),$$

(6.12)

$$\rho_{\phi(j+1)+1,k,2}(\lambda) = (-i)^{(\alpha_{\phi(j+1)+1}+1)/2+k-1} \cdot \left[\sin\left(\frac{\theta_{j+1,2}}{2}\right)\right]^{-1}$$

$$\cdot i \operatorname{Im} \exp\left\{ i \frac{\theta_{j+1,2}}{2} \left[\tfrac{1}{2}(\alpha_{\phi(j+1)+1} + 1) + k - 1 \right] \right\}$$

$$+ O(\lambda^{-1/(\alpha_{\phi(j+1)+1}+2)});$$

and both $p_2^j(\lambda)$, $q_2^j(\lambda)$ *in* (6.3) *are equal to*

(6.13) $$K(\alpha_{\phi(j+1)}, \alpha_{\phi(j+1)+1}) + O(\lambda^{-1/\sigma(j+1)}),$$

where $K(\alpha_{\phi(j+1)}, \alpha_{\phi(j+1)+1})$ *is one nonzero constant which depends only on* $\alpha_{\phi(j+1)}$ *and* $\alpha_{\phi(j+1)+1}$, *and*

$$\sigma(j+1) = \max\{\alpha_{\phi(j+1)}, \alpha_{\phi(j+1)+1}\} + 2.$$

PROOF. As in the last lemma $\rho_{\phi(j+1),2,2} = \tilde{c}_{(\alpha_{\phi(j+1)}+1)/2}$ where $\tilde{c}_k$, $k = 0, 1, \ldots, (\alpha_{\phi(j+1)} + 1)/2$, satisfy the recursive relations

(6.14)

$$\tilde{c}_0 = 1, \quad \tilde{c}_1 = -2\cos\left(\frac{\theta_{j+1,1}}{2}\right)i + O(\lambda^{-1/(\alpha_{\phi(j+1)}+2)})$$

$$\tilde{c}_{k+2} = -2\cos\left(\frac{\theta_{j+1,1}}{2}\right)i\,\tilde{c}_{k+1} + \tilde{c}_k + O(\lambda^{-1/(\alpha_{\phi(j+1)}+2)}).$$

By a formula analogous to (6.11) for $k = (\alpha_{\phi(j+1)} + 1)/2$, we deduce the formula for $\rho_{\phi(j+1),2,2}$ in (6.12). Checking the entries for $[N_{\phi(j+1)}]^k$, $k = 0, 1, \ldots,$ one sees that $\rho_{\phi(j+1),2,1}$ should simply be $\tilde{c}_{(\alpha_{\phi(j+1)}+1)/2-1}$. Thus, by (6.11) again we deduce the formula for $\rho_{\phi(j+1),2,1}$ in (6.12). The second line of (6.12) is derived in the same manner. (6.13) is derived from (6.3) and (6.12), using the fact that, for $r = 1, 2$,

(6.15)

$$\frac{\theta_{j+1,r}}{2}\left[1 + \frac{1}{2}(\alpha_{\phi(j+1)+r-1} + 1)\right] = \frac{\pi}{2} + \frac{\pi}{2}(\alpha_{\phi(j+1)+r-1} + 2)^{-1},$$

$$\frac{\theta_{j+1,r}}{2}\,\frac{1}{2}(\alpha_{\phi(j+1)+r-1} + 1) = \frac{\pi}{2} - \frac{\pi}{2}(\alpha_{\phi(j+1)+r-1} + 2)^{-1}.$$

(6.15) is used to simplify the products, and to deduce that the leading constant in both p_2^j and q_2^j are the same.

PROOF OF THEOREM 1.1. The solution of (3.1) in the first column of $Z_{12}^1(x, \lambda)$ is subdominant as $x \to +\infty$, and the second column tends to infinity as $x \to +\infty$. On the other hand, the second column of $Z_{21}^n(x, \lambda)$ is subdominant as $x \to -\infty$, and its first column tends to infinity as $x \to -\infty$. In view of Theorem 5.1, equation (5.22), λ in the region (3.4) will be an eigenvalue to problem (1.1), (1.2) iff the (2, 2)th entry of $T_1\Pi_{j=1}^{m-1}R_jT_{j+1}(\lambda)$ is exactly zero.

Let λ be such an eigenvalue and let y be the corresponding solution of (1.1). If we multiply (1.1) by $\bar{y}$ (conjugate of y) and integrate from $-\infty$ to $+\infty$, we obtain

$$\int_{-\infty}^{+\infty} \bar{y}y''\,dx - \lambda^2\int_{-\infty}^{+\infty}|y|^2p\,dx = 0.$$

Integrating the first term by parts, and using the form of the solution and its derivative by means of Lemma 3.1, 3.2, as $x \to \pm\infty$, we obtain

$$-\int_{-\infty}^{+\infty}|y'|^2\,dx - \lambda^2\int_{-\infty}^{+\infty}|y|^2p\,dx = 0.$$

Since $y' \not\equiv 0$, $\int_{-\infty}^{+\infty}|y|^2p\,dx \neq 0$ and λ^2 is real. Therefore, if λ is in (3.4) or S, of large modulus, then λ is real. This proves part (i).

Let the (2, 2)th entry of $T_1(\lambda)\Pi_{j=1}^{m-1}[R_j(\lambda)T_{j+1}(\lambda)]$, for λ in region (3.4), be denoted by $f(\lambda)$. Using Lemmas 5.4, 5.5 and 6.1, 6.2, one derives that

$$f(\lambda) = h(\lambda)\left[v_2^0(\lambda)\prod_{j=1}^{m-1} v_2^j(\lambda) + \mu(\lambda)\right] \tag{6.16}$$

where $h(\lambda)$ is a product of nonzero constants and exponential factors which are never zero, and $v_2^0(\lambda)$ denotes the (2, 2)th entry of $T_1(\lambda)$, which is of the form (5.10) or (6.8). Therefore all eigenvalues of large modulus in region (3.4) must satisfy the equation

$$\prod_{j=0}^{m-1} v_2^j(\lambda) = \mu(\lambda). \tag{6.17}$$

If G_{r+1} has one turning point, then by formulas (6.5) and (6.8) one sees that $v_2^r(\lambda)$ tends to a nonzero constant as $\lambda \to \infty$ in region (3.4). Formula (6.17) is thus equivalent to

$$\prod_{\substack{j=0\\ j\in J_0}}^{m-1} v_2^j(\lambda) = \mu(\lambda) \tag{6.18}$$

where $J_0 = \{j \mid G_{j+1}$ has two turning points$\}$. Lemmas 6.1 and 6.4 show that such $v_2^j(\lambda)$ is of the form

$$K(\alpha_{\phi(j+1)}, \alpha_{\phi(j+1)+1})$$
$$\cdot\left[1 + \exp\{2\lambda\xi(a_{\phi(j+1)+1}, a_{\phi(j+1)})\} + O(\lambda^{-1/\sigma(j+1)})\right]\left[1 + O(\lambda^{-1})\right].$$

Therefore (6.18) can be written in the form (1.3). This proves part (ii).

If J_0 is empty (i.e. $r = 0$), the last paragraph implies that equation (6.17) has no solution of large modulus in region (3.4). Hence, there is no large eigenvalue there. If J_0 is nonempty (i.e. $r \neq 0$), we can readily assert the existence of zeroes, for equation (6.18) or (1.3), of large modulus in S. These are eigenvalues for problems (1.1), (1.2). Equation (1.3) implies that all eigenvalues in S of large modulus must satisfy

$$\prod_{\substack{k=1\\k\in J}}^{n-1}\left[1 + \exp\left\{2\lambda\int_{a_{k+1}}^{a_k}\sqrt{p(x)}\,dx\right\}\right] = G(\lambda) \tag{6.19}$$

where $G(\lambda)$ is analytic in λ, satisfying $G(\lambda) = O(|\lambda|^{-\theta})$ as $\lambda \to \infty$ in S. Let $\varepsilon > 0$ be any small positive constant. Let

$$\eta_{k\nu} = \frac{\left(\nu - \frac{1}{2}\right)\pi}{\int_{a_{k+1}}^{a_k}\sqrt{|p(x)|}\,dx}$$

$k \in J$, $\nu = 1, 2, \ldots$, be r sequences of roots for the equation

$$\prod_{\substack{k=1\\k\in J}}^{n-1}\left[1 + \exp\left\{2\lambda\int_{a_{k+1}}^{a_k}\sqrt{p(x)}\,dx\right\}\right] = 0. \tag{6.20}$$

With each of the points $\eta_{k\nu}$ on the λ plane as center, draw a circle of radius $K_1\eta_{k\nu}^{(-\theta/r)+\varepsilon}$, where K_1 is a fixed positive constant. Let $\mathfrak{T}$ be the union of the interiors of all these circles, and $\mathfrak{T}_{k\nu}$ be its component containing the point $\eta_{k\nu}$. Because of the magnitudes of the radii of the circles, and the fact that there are r factors in (6.20), one readily concludes that for λ at the boundaries of $\mathfrak{T}_{k\nu}$, $\nu = 1, 2, \ldots$, we have the inequality

$$\left|\prod_{\substack{k=1\\k\in J}}^{n-1}\left[1 + \exp\left\{2\lambda\int_{a_{k+1}}^{a_k}\sqrt{p(x)}\,dx\right\}\right]\right| > K_2\left[|\lambda|^{(-\theta/r)+\varepsilon}\right]^r = K_2|\lambda|^{-\theta+\varepsilon r},$$

where K_2 is a positive constant. Consequently, for λ of large modulus at the boundary of $\mathfrak{T}$, one has

$$|G(\lambda)| < \left|\prod_{\substack{k=1\\k\in J}}^{n-1}\left[1 + \exp\left\{2\lambda\int_{a_{k+1}}^{a_k}\sqrt{p(x)}\,dx\right\}\right]\right|.$$

By means of the general form of Rouché's Theorem, we conclude that equation (6.19) has exactly the same number of roots in $\mathfrak{T}_{k\nu}$ as equation (6.20), for ν sufficiently large. Further, for large enough λ, these are all the roots for (6.19). This proves part (iii).

REMARK. The cases when hypotheses are less restrictive than that of (H2a) can also be investigated by analogous techniques. The author will study such situations shortly. The author wishes to thank Professor Y. Sibuya for stimulating conversations on the subject matter.

REFERENCES

1. M. A. Evgrafov and M. V. Fedoryuk, *Asymptotic behaviour of solutions of the equation* $w''(z) - p(z, \lambda)w(z) = 0$ *as* $\lambda \to \infty$ *in the complex z-plane*, Uspehi Mat. Nauk **21** (1966), no. 1(127), 3–50 = Russian Math. Surveys **21** (1966), no. 1, 1–48. MR **35** #459.

2. A. W. K. Leung, *Connection formulas for asymptotic solutions of second order turning points in unbounded domains*, SIAM J. Math. Anal. **4** (1973), 89–103. MR **48** #11707.

3. ———, *Lateral connections for asymptotic solutions around higher order turning points*, J. Math. Anal. Appl. **50** (1975), 560–578.

4. Y. Sibuya, *Subdominant solutions of the differential equation* $y'' - \lambda^2(x - a_1)(x - a_2) \cdots (x - a_m)y = 0$, Acta. Math. **119** (1967), 235–272. MR **37** #529.

5. W. Wasow, *Asymptotic expansions for ordinary differential equations*, Interscience, New York, 1965. MR **34** #3041.

6. Lawrence Weinberg, *The asymptotic distribution of eigenvalues for the boundary value problem* $y''(x) - \lambda^2 p(x)y(x) = 0, y \in L_2(-\infty, +\infty)$, SIAM J. Math. Anal. **2** (1971), 546–566. MR **46** #5778.

7. A. Leung, *Errata*: "*Connection formulas for asymptotic solutions of second order turning points in unbounded domains*" (SIAM J. Math. Anal. **4** (1973), 89–103) MR **48** #11707), SIAM J. Math. Anal. **6** (1975), 600. MR **50** #13778.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CINCINNATI, CINCINNATI, OHIO 45221